

The several dimensional gambler's ruin problem

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Abstract

We derive asymptotic expressions for the simple N -dimensional random walk with perspective to the duration of play in the multidimensional gambler's ruin problem. We show that, under suitable rescalings, all p -moments of exit-times from balls in the L -infinity metric, and all p -moments of partial-maxima values in this metric, possess associated asymptotic limits, admitting two representations each. The proofs lean upon deriving certain multidimensional extensions of the Erdős-Kac limit theorem, which to this end we revisit.

Key-words: Weak limit laws; Brownian motion; Invariance principle; Convergence of moments; Exit times; Running maxima; Erdős-Kac theorem; Laplace transforms; Uniform integrability; Boundary value problems

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1 Introduction

1.1 Motivation: fair games of chance

Originating from correspondence of Blaise Pascal and Pierre de Fermat in 1656, the ‘gambler's ruin problem’ regards the game of two players engaging in a series of independent and identical bets up until one of them is bankrupt, viz. ruined.¹ The general ‘gambler's ruin formula’, which regards the chances of each player winning, was shown by Abraham De Moivre in 1712. The solution to the problem of the ‘duration of play’, which regards a ‘time-limited extension’ of the said formula,

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¹The first formulation of the gambler's ruin problem had always been credited to work of Huygens in 1657, only because his correspondence, which mentions his source, was not published until 1888. For more on the historical background to this problem and its time-limited extension, we refer for instance to the notes in [§ 7.5, Ethier [E10]] and the references therein.

also dates back to 1712 and is due to De Moivre.² Different formulae for this were obtained afterward by Montmort, Nicolaus Bernoulli, as well as Joseph-Louis Lagrange.³ Regarding the fair bets case it is a celebrated result that the expected value of the duration of play equals the product of the initial fortunes of the players.⁴ Some of the original motivation for this work may be sought into study of generalizations of this formula in the higher-dimensional setting, which we describe next.

The ‘fair gamblers’ ruin problem’ may be cast as a simple betting game in the following settings, which extend the classical one in a natural way. In one of the interpretations, two players are both in possession of initial fortunes in a number of more than one currencies. At every round of this game a fair bet takes place. The winner of the bet receives a payoff which amounts to one monetary unit of a currency chosen independently in an even way among all currencies. The game continues in independent rounds and is over as soon as either one of the players runs out of any currency.

This problem may also be interpreted in the setting of a gambling competition between two teams with the same number of players. In this interpretation, opponents are matched into pairs in the outset. Every round of this game then consists of choosing evenly one of the pairs. The chosen pair of players then bets on a fair game and the winner receives one monetary unit from its opponent. The game carries on in this fashion for as long as none of the players of either team is bankrupt. Note that, since all bets are fair and the probability of winning for either player in the former interpretation, or for either team in the latter one, is obviously the same, the sole quantity of interest in the study of this game is the duration of its play.

Nonetheless the aforementioned elegant and simple expression for the expected value of the duration of play in the one-dimensional fair gambler’s ruin problem, concerning the more general settings described, in the words of Orr and Zeilberger [OZ94] ‘no closed-form solution of this problem is known to exist, and probably none does exist’.⁵ Prior to presenting our results regarding asymptotics for all moments of the duration of play in the fair gamblers’ ruin problem, a brief review of the long mathematical history that the intimately associated ‘absorption problem’ enjoys is given.

1.2 The Absorption Problem

The ‘simple one-dimensional random walk’ is the celebrated discrete-time stochastic process comprising the sequence of successive partial sums of a sequence of independent uniformly distributed $\{-1, 1\}$ -valued random variables, which may be thought of as modeling the motion of a particle on \mathbb{Z} jumping at every instance of time to either one of its nearest-neighbors sites according to the outcomes of this sequence.

²A derivation of this formula may be found in [Feller [Fl68], Chpt. XIV, § 5], where the technique of expanding rational functions in partial fractions is employed.

³see (7.38) and (7.164-5) respectively in [E10].

⁴A derivation of this via Doob’s Optional-Stopping Theorem may be found in any standard textbook in probability dealing with martingales.

⁵Something which is also in accord with the, out of his own experience, belief of the author.

The ‘absorption problem’ regards the asymptotic law of the times that the modulus of this walk attains new maxima values, under appropriate rescaling; note also that these times correspond to the particle's ‘exit-times’ from symmetric intervals about the origin.⁶ Note that, in principle this asymptotic law may be derived from any of the aforementioned solutions to the problem of the duration of play.

Spitzer [Chpt. V, [Sp76]] provides with an original review to the absorption problem, pointing out to a list of earlier treatments for completeness⁷. Regarding arbitrary zero-mean (positive and finite variance) increment-distributions random walks, the rigorous solution to this problem is due to Erdős and Kac [Theorem II, [EK46]], although the asymptotic distribution was indeed already known in the beginnings of the previous century to Bachelier who worked in the context of Brownian motion. The crux of the Erdős and Kac solution to the general problem is that, for all zero-mean increment-distributions, the limiting distribution must be identical, up to a scaling constant which only depends on the variance of the increment-distribution. The study of random walks is intimately associated to the celebrated continuous-time stochastic process refer to as Brownian motion, a.k.a. the Wiener process. Indeed, the key idea of the Erdős and Kac solution paved the way to establishing the deep connection amongst random walks and Brownian motion, known as the ‘invariance principle’, a.k.a. the ‘functional central limit theorem’, or ‘Donsker's theorem’, due to Donsker [Dn51].⁸

The mathematically rigorous theory of stochastic processes in continuous-time was initiated with the seminal work by Kolmogorov [Kl31]. A rigorous solution of the associated absorption problem for Brownian motion was made available via the celebrated ‘Lévy's triple law’, which regards the joint probability distribution of the Brownian motion at fixed times together with its running minimum and maximum, and is derived by Lévy [Lv48].⁹ The closely related distribution of the running-maxima of the modulus of the Brownian Bridge, which corresponds to the distribution of the error term in the Kolmogorov-Smirnov statistic, was derived by Kolmogorov [Kl33].¹⁰

A different, elementary approach for deriving a time-limited gambler's ruin expression in the fair bets case is shown in [§ 21, [Sp76]].¹¹ This expression allows by means of a limiting procedure the derivation of a solution to the absorption problem, cf. [§ 21, Proposition 5, [Sp76]], which by appeal to the invariance principle yields the corresponding result in the general case of arbitrary zero-mean increment-distribution random walks. Nevertheless, relying on the method of moments, an ex-

⁶Regarding the problem's nomenclature, we note that it derives from the equivalent perspective of the asymptotic law of the duration of the motion of the particle in finite intervals with absorbing endpoints, as their length tends to infinity.

⁷cf. [footnote 1, p. 237, [Sp76]]

⁸Note that the equivalence among the before-mentioned absorption problems for arbitrary zero-mean increment-distribution random walks and for the Brownian motion may be justified by an instance of this theorem, see [(13), Theorem 3] below for a precise formulation.

⁹For a justification of this claim we refer to [footnote 1, p. 169, Schilling and Partzsch [SR14]]; cf. also with Remark 5 below for pointers to the literature regarding its proof.

¹⁰For a derivation of this we refer to [Feller [Fl70], Chpt. I, § 12].

¹¹The technique there relies on generalizations of spectral representations for transition matrices and orthogonal polynomials, and is different from those mentioned above; see, for instance, [§ 5, Chpt. 10, Karlin and Taylor [KT81]] for an introduction to this approach.

tension of this approach to cover the general case is in addition offered in [§§ 22, 23, [Sp76]]; this approach does not only rely on technically challenging and novel methods of intrinsic interest, but also provides with various additional results, interesting in their own right.

1.3 The multidimensional case

The simple N -dimensional random walk is the basic discrete-time stochastic process modeling the motion of a particle in \mathbb{R}^N composed by unit-length displacements in each of the cardinal directions with equal probability, independently at every instance of time.¹² The analysis and understanding of random walks has been the epicenter of the theory of probability since the very inception of the subject, and is therefore one of its most exhaustively studied topics. For the classic reference devoted to the random walk, we refer to Spitzer [Sp76]; for more recent accounts, we refer to Lawler and Limic [LL10] and Révész [R90]. For introductory treatments devoted to the random walk, we refer to Lawler [Lw10] and Lesigne [Ls05]. For resources on background probability material, including excellent accounts on basic random walk theory, we refer to Billingsley [Bl68], Breiman [Br92], and Feller [Fl68, Fl70], whose everlasting influences cannot be overestimated. For more recent treatments in this regard, we refer to Durrett [Dr10], Gut [G12, G09], Kallenberg [Kll97], Stroock [St05], and Williams [W91]. For accounts laying emphasis to Brownian Motion, we refer to Mörters and Peres [MP10], Revuz and Yor [RY94], Rogers and Williams [RW93], and Schilling and Partzsch [SR14].

As in the 1-dimensional case, asymptotics for the duration of play in the fair gamblers' ruin problem correspond to multidimensional absorption problems or, in other words, to multidimensional exit-times asymptotics. Further, it is easy to see that, in the case of equal initial fortunes, the duration of play in the fair gamblers' ruin problem corresponds to exit-times of the N -dimensional walk from hypercubes, which is, L^∞ -balls about the origin. Note that the expected values of exit-times from L^2 -balls of radius r about the origin scaled with r^2 tends to 1, as $r \rightarrow \infty$, irregardless of N .¹³

Theorem 1 is our main result and is formally stated in Section 2 next. Its first part provides with asymptotics for all p -moments of the duration of play in N -dimensions, which in contrast turn out to depend on the dimension N . In addition, the second part of Theorem 1 provides with asymptotic limits for all p -moments of partial-maxima values of the N -dimensional walk in the L^∞ -metric under appropriate rescaling. These two asymptotic limits turn out to be associated, which is an aftereffect of the fact that the two random sequences are inverses of one another, in that exit times correspond to instances that new partial-maxima values are attained in this metric. In this sense Theorem 1 makes the resultant limit interconnection precise. The technique of proof of Theorem 1 is elementary probabilistic with a per-

¹²More formally this may be put as considering the successive partial-sums of uniformly distributed $\mathcal{B}_N \cup -\mathcal{B}_N$ -valued random vectors, where \mathcal{B}_N is the standard orthonormal basis of unit vectors of \mathbb{Z}^N and $-\mathcal{B}_N$ are their negative-signed counterparts; cf. with (1) for precise definition.

¹³One can easily show that indeed this expected value lies in $[r^2, (r+1)^2]$ by a straightforward extension of the application of Doob's Optional-Stopping Theorem we mentioned in footnote 3; cf., for instance, with [§ 1.4.2, [Lw10]] for an explicit computation.

spective to studies of the random walk in conjunction with the Brownian motion, and indeed Proposition 2 regarding the latter is key to its proof.

We note that the former-mentioned asymptotics in the first part of Theorem 1 extend those derived by Kmet and Petkovšek [KP02], that deal with expected values (1-moments) by means of discrete Fourier methods for solving the associated Poisson partial differential equation. The expressions we derive in this case are also contrasted and found to be simpler alternatives to the general-dimension asymptotics in [KP02], mostly in that our formulae involve 1-fold instead of N -fold sums (cf. Remark 2). Our main results we made mention of are formally presented in Section 2 next.

In addition, in Section 3 we revisit the version of the Erdős and Kac theorem stated in Theorem 3. Subsection 3.1 comprises a related observation and a simple consequence of this theorem, which we invoke later. In Subsection 3.3 we show an elementary path, which to the limits of our knowledge is not pursued elsewhere, to prove this Theorem for completeness, whereas in Subsection 3.2 we give a miscellany of other available proof approaches to this theorem from the literature.

2 Main results

The simple N -dimensional random walk $(\mathbf{Z}_t : t \geq 0)$ is defined via $(\boldsymbol{\omega}_s : s \geq 1)$, a collection of independent random variables with identical distribution, given as follows:

$$\mathbb{P}(\boldsymbol{\omega}_s = \mathbf{w}) = \frac{1}{2N}, \quad \text{for } \mathbf{w} = \pm \mathbf{e}_i, \quad i = 1, \dots, N,$$

where \mathbf{e}_i is the N -dimensional vector whose i th component is 1 and others are 0¹⁴. We then let

$$\mathbf{Z}_t = \sum_{s=1}^t \boldsymbol{\omega}_s. \quad (1)$$

We may equivalently define (\mathbf{Z}_t) as the time-homogeneous Markov chain with state-space \mathbb{Z}^N and transition probabilities given by,

$$\mathbb{P}(\mathbf{Z}_t = \mathbf{z} \mid \mathbf{Z}_{t-1} = \mathbf{y}) = \frac{1}{2N}, \quad (2)$$

for $\mathbf{z} - \mathbf{y} = \{\pm \mathbf{e}_i, i = 1, \dots, N\}$, and $\mathbf{Z}_0 = \mathbf{0}$.

To state our main result, let

$$\widetilde{T}_r = \min\{t : \mathbf{Z}_t \notin \mathcal{B}_r\} \quad (3)$$

where $\mathcal{B}_r = \{\mathbf{z} \in \mathbb{Z}^N : |\mathbf{z}| \leq r\}$, and $|\cdot|$ denotes the L -infinity norm¹⁵. Further, let

$$\widetilde{M}_t = \max_{1 \leq s \leq t} |\mathbf{Z}_s|. \quad (4)$$

¹⁴so that $\mathcal{B}_N = (\mathbf{e}_i)_{i=1, \dots, N}$, mentioned above, is the standard orthonormal basis of \mathbb{R}^N .

¹⁵Recall that $|\mathbf{z}| = \max\{|z_1|, |z_2|, \dots, |z_N|\}$, $\mathbf{z} = (z_1, \dots, z_N)$, so that \mathcal{B}_r is the hypercube with vertices $(\pm r, \dots, \pm r)$, a.k.a. the Moore neighborhood range $r \geq 1$.

Theorem 1. *We have that*

$$\frac{1}{r^{2p}} \mathbb{E} \tilde{T}_r^p \rightarrow \mathbb{E} \tilde{T}_N^p \quad \text{as } r \rightarrow \infty, \quad (5)$$

and further, that

$$\frac{1}{t^{p/2}} \mathbb{E} \tilde{M}_t^p \rightarrow \mathbb{E} \tilde{M}_N^p \quad \text{as } t \rightarrow \infty, \quad (6)$$

$p \geq 1$, $\tilde{F}_{N,p}(u) = \mathbb{P}(\tilde{T}_N^p \leq u) = \tilde{F}_N(u^{1/p})$ and $\tilde{G}_{N,p}(v) = \mathbb{P}(\tilde{M}_N^p \leq v) = \tilde{G}_N(v^{1/p})$, and where

$$\tilde{F}_N(t) = 1 - \left(H \left(\frac{t}{N^2} \right) \right)^N, \quad (7)$$

and

$$\tilde{G}_N(x) = \left(H \left(\frac{1}{N^2 x^2} \right) \right)^N, \quad (8)$$

where also

$$H(y) = \frac{4}{\pi} \sum_{n \geq 0} \frac{(-1)^n}{2n+1} \cdot \exp \left(-\frac{\pi^2}{8} (2n+1)^2 y \right), \quad y > 0. \quad (9)$$

Remark 1. The function $H(\cdot)$ can be alternatively expressed as follows.

$$H(y) = \int_{-1/\sqrt{y}}^{1/\sqrt{y}} \sum_{k=-\infty}^{\infty} (-1)^k \exp \left(-\frac{1}{2} \left(t + \frac{2k}{\sqrt{y}} \right)^2 \right) dt. \quad (10)$$

For practical purposes, (9) is more useful for large values of y , whereas (10) converges faster only for small values of y .¹⁶ For more on (9), (10) and their equivalence, see Subsection 3.2 below.

Remark 2. The methods in [KP02] yield explicit expressions regarding $\mathbb{E} \tilde{T}_N$, for all N , in [[KP02], § 5, Theorem 2], and also yield an estimate for the rate of this convergence in the $N = 2$ case [[KP02], § 5, Theorem 1]. Their general formulae in our notation is as follows.

$$\mathbb{E} \tilde{T}_N = N \left(1 - \frac{2^{2N+1}}{\pi^{N+1}} \sum_{k_1, k_2, \dots, k_N \geq 0} \frac{(-1)^{\sum_{j=1}^{N-1} k_j} \prod_{j=1}^{N-1} (1/(2k_j + 1)) \sum_{j=1}^{N-1} (1/(2k_j + 1)^2)}{\cosh(\frac{\pi}{2}) \sqrt{\sum_{j=1}^{N-1} (2k_j + 1)^2}} \right).$$

On the other hand, [(5), Theorem 1] gives directly by standard moment expressions for positive random variables that, for all $p \geq 1$,

$$\begin{aligned} \mathbb{E} \tilde{T}_N^p &= \int_0^\infty \left(H \left(\frac{t^{1/p}}{N^2} \right) \right)^N dt, \\ &= p \int_0^\infty t^{p-1} \left(H \left(\frac{t}{N^2} \right) \right)^N dt, \end{aligned}$$

where H is as in (9), or as in (10).

To see the connection between the statement given next and Theorem 1, note that the covariance matrix of (\mathbf{Z}_t) is $\Sigma := \mathbb{E}(\mathbf{Z}_1 \mathbf{Z}_1^T) = \frac{1}{N} I$, where I is the identity matrix.

¹⁶cf. with, for instance, [Remark 1, p. 21, Révész, [R90]].

Proposition 2. Let $\widetilde{\mathbf{W}}_s$ be N -dimensional Brownian motion with covariance matrix $\frac{1}{N}I$. Let $\widetilde{T}_N = \inf\{s : \widetilde{\mathbf{W}}_s \notin \mathbf{B}_1\}$ and let also $\widetilde{M}_N = \sup_{0 \leq s \leq 1} |\widetilde{\mathbf{W}}_s|$, where $\mathbf{B}_1 = \{\mathbf{x} \in \mathbb{R}^N : |\mathbf{x}| \leq 1\}$. Let also $\widetilde{F}_N(\cdot)$ and $\widetilde{G}_N(\cdot)$ be as in (7) and (8) respectively. We have that

$$\mathbb{P}(\widetilde{T}_N < t) = \widetilde{F}_N(t), \quad (11)$$

and that

$$\mathbb{P}(\widetilde{M}_N < x) = \widetilde{G}_N(x). \quad (12)$$

A key preliminary here is the celebrated Erdős and Kac theorem, a version of which that is apt for our purposes is stated next.

Theorem 3 (Erdős and Kac [EK46]). Let $S_t = \sum_{s=1}^t \xi_s$, where $(\xi_i; i \geq 1)$ are i.i.d. random variables such that $\mathbb{E}(\xi_i) = 0$ and that $\mathbb{E}(\xi_i^2) = 1$. Let $\tau_b = \inf\{t : |S_t| \geq b\}$ and also let $m_t = \max\{|S_i| : i \leq t\}$. Further, let $(W(t) : t \in \mathbb{R}_+)$ be standard linear Brownian motion and let $T = \inf\{t : |W(t)| = 1\}$ and also let $M = \sup_{t \in [0,1]} |W(t)|$. We have that

$$\frac{\tau_b}{b^2} \xrightarrow{d} T \quad \text{as } b \rightarrow \infty, \quad (13)$$

and that,

$$\frac{1}{\sqrt{t}} m_t \xrightarrow{d} M \quad \text{as } t \rightarrow \infty. \quad (14)$$

Further, if $\Phi(t) := \mathbb{P}(T < t)$ and $\Gamma(x) := \mathbb{P}(M < x)$, then we have that

$$\Phi(t) = 1 - H(t), \quad (15)$$

and that

$$\Gamma(x) = H\left(\frac{1}{x^2}\right), \quad (16)$$

where $H(\cdot)$ is as in (9), or (10).

The remainder comprises Sections 3 and 4. The contents and organization of these sections are outlined as follows.

In Section 4 we give the proofs of Theorem 1 and Proposition 2 stated above. The proof of Theorem 1 relies on first showing convergence in distribution analogues of [(5), Theorem 1] and of [(6), Theorem 1], derived from Proposition 2 combined with applications of the multidimensional functional central limit theorem, carried out in Lemma 18. To extend convergence in distribution to the convergence of moments in Theorem 1, we prove in Proposition 16 uniform integrability of the sequences of random variables in (5) and (6). The proof of this Proposition is carried out in two steps and requires some preparatory work, which we take upon in Subsection 4.1.

The first step comprises of proving this Proposition in dimension $N = 1$, and is done in Lemma 14. We note that, regarding 1-dimensional, arbitrary zero-mean increment-distribution random walks, [Theorem 1, (5)] is already shown in [§ 23, Spitzer [Sp76]], combine Propositions 3 and 6 there. Our proof of uniform integrability of the sequence in (5) in dimension $N = 1$ is based upon this result, and is done in [(34), Lemma 14]. Our proof of uniform integrability of the sequence in (6) in dimension $N = 1$ uses a maximal inequality, and is done in [(35), Lemma 14].

The second step for deriving this Proposition comprises of the couplings in Lemma 13. These couplings allow to extend the uniform integrability results in Lemma 14 to N dimensions. These preparatory Lemmas 13 and 14 comprise Subsection 4.1.

As noted already, the interconnection among [(7), Theorem 1] and [(8), Theorem 1] is due to that exit-times are times that new partial-maxima values are achieved for the random walk. The fact that these two limit theorems may be associated by coupling is already pointed out and exploited in the context of 1-dimensional random walks in [Theorem 3, § 23, [Sp76]]. We note that our proof approach in Proposition 2 is facilitated by exploiting the corresponding coupling connection directly for the Brownian motion limiting objects. Further, we note that the method of deriving Proposition 4, refer to next, is also an instance of this approach in dimension one.

In Section 3, we revisit Theorem 3. In Subsection 3.1, we observe that two different routes for proving [(15), Theorem 3] and [(16), Theorem 3] are possible, due to their interconnection which we point out to in Proposition 4 there. An easy consequence of Theorem 3 combined with Proposition 4 is also derived there in Corollary 6. This corollary is used later in the proof of Proposition 2. By means of Proposition 4, various different routes for deriving Theorem 3 are made available. We collect them together, along with various associated pointers to the literature, in Subsection 3.2.

In Subsection 3.3, we show an elementary proof approach to Theorem 3 regarding zero-mean increment-distribution random walks. The first statement we give there is two-fold and provides, in [(19), Theorem 7], with the convergence of Laplace transforms corresponding to [(13), Theorem 3], and, in [(20), Theorem 7] with the convergence of Laplace transforms corresponding to the so-called first-passage times. We derive a proof of the Theorem 3, from [(19), Theorem 7] in conjunction with Proposition 4 by invoking some known facts from [10, p. 273, [Sp76]]. Further, we derive in Corollary 8 the limit law of first-passage times as another direct byproduct. In this way, our approach brings together Theorem 3 with the celebrated stable law exponent $1/2$ for first passage times. The remainder of Subsection 3.3 is then devoted to devising an elementary proof of (both parts of) Theorem 7, which to our knowledge is not developed elsewhere (cf. Remark 7), in order for our approach there to be elementary in its entirety. This proof relies on the invariance principle and comprises of deriving the associated Laplace transforms in the simple random walk case, by building upon variants of known arguments relying on Doob's Optional Stopping Theorem, cf. [§ 10.12, Williams [W91]], along with a simplifying detour via Lemma 11, which in effect follows known arguments, cf. for instance, [Corollary 2.17, Kallenberg [Kll97]]. By Lemma 12, which extends basic calculus results suggested in [§ 1.3, Lalley [Ll]], the asymptotics of both these Laplace transforms are then derived there.

3 The Erdős and Kac theorem revisited

3.1 A closely related observation and a consequence of it

Let $\Phi(\cdot)$ and $\Gamma(\cdot)$ be as in Theorem 3; these two distributions are associated as follows.

Proposition 4. $\Phi(t) = 1 - \Gamma\left(\frac{1}{\sqrt{t}}\right)$, $t > 0$.

Proof. Let $M(t) = \sup_{s \in [0, t]} |W(s)|$ and observe that by coupling the following holds

$$\{T < t\} = \{M(t) > 1\}. \quad (17)$$

However, from the Brownian scaling property $W(s) \stackrel{d}{=} cW(s/c^2)$, for all $c > 0$, we have the following.

Lemma 5. $M(t) \stackrel{d}{=} \sqrt{t}M(1)$.

Proof of Lemma 5. By Brownian scaling

$$\begin{aligned} M(t) &\stackrel{d}{=} c \sup_{s \in [0, t]} |W(s/c^2)| \\ &\stackrel{d}{=} cM(t/c^2), \end{aligned}$$

plugging $c = \sqrt{t}$ in the display above proves the statement. \square

Note that (17) together with Lemma 5 give

$$\begin{aligned} \Phi(t) := \mathbb{P}(T < t) &= \mathbb{P}(M(t) > 1) \\ &= \mathbb{P}\left(M(1) > \frac{1}{\sqrt{t}}\right) \\ &= 1 - \Gamma\left(\frac{1}{\sqrt{t}}\right), \end{aligned} \quad (18)$$

where in (18) we used continuity of $\Gamma(\cdot)$. The proof is thus complete. \square

For ease of reference below, we record the following consequence of Theorem 3 by Proposition 4.

Corollary 6. Let $M(t) = \sup_{s \in [0, t]} |W(s)|$ and let $\Gamma_t(x) = \mathbb{P}(M(t) < x)$. We have that

$$\Gamma_t(x) = \Gamma\left(\frac{x}{\sqrt{t}}\right) = H\left(\frac{t}{x^2}\right),$$

where H is as in (9), or (10).

Proof. This follows from Lemma 5 above combined with [(16), Theorem 3]. \square

3.2 A miscellany of proof approaches to Theorem 3

In this section we comment and give pointers to the literature regarding various possible proof approaches to Theorem 3. Limit theorems [(13), Theorem 3] and [(14), Theorem 3] follow by applications of Donsker's Theorem (cf. Lemma 18, in the proof of Theorem 1 below, which shows their higher-dimensional analogues). Hence, we focus here on the remaining parts of the statement, [(15), Theorem 3] and [(16), Theorem 3]. Remarks 3, 4, and 5 regard proof approaches to (15) and to (16), which we note that are interconnected via Proposition 4. In Remark 6, we comment on the equivalence of $H(\cdot)$ as in (10) and as in (9).

Remark 3. Regarding deriving (15) for $H(\cdot)$ as in (9), one can follow [§ 21, Proposition 5, [Sp76]], which we commented upon in the last paragraph of § 1.2, along with (13).

Remark 4. An approach leading to (16) for $H(\cdot)$ as in (9) is through the Feynman-Kac formulas for Brownian motion and showing a solution of the heat equation by use of the separation of variables technique; cf. the argument following [Theorem 7.45, Mörters and Peres [MP10]].

Remark 5. Various proofs of Lévy's triple law, which we mentioned in the second paragraph of § 1.2 and which lead to (16) for $H(\cdot)$ as in (10), are follows. One approach goes through first deriving the associated law for the simple random walk (either by induction, or the technique of repeated reflections, aka method of images), and then using the classic central limit theorem and Donsker's Theorem; cf. [Chpt. 2, § 11.1, Billingsley [Bl68]]. A closely related proof approach goes through arguments employing the said technique directly for standard Brownian motion instead, see for instance, [Theorem 8.7.3, Durrett [Dr10]], or [§ 6.5, Schilling and Partzsch [SR14]]. A different approach goes through the Feynman-Kac formulas for stochastic integrals and a method for verifying a solution to the heat equation, which may be motivated by heuristics, cf. [Theorem 7.45, Mörters and Peres [MP10]].

Remark 6. For showing the equivalence of $H(\cdot)$ as in (10) with that in (9), one can combine the argument in Remark 4 with Lévy's triple law, for the various routes to which we refer to in Remark 5. Alternatively, through Proposition 4, one can employ the statement we infer Remark 3, along with Lévy's triple law. It is also possible to transform $H(\cdot)$ as in (10) to $H(\cdot)$ as in (9), see, for instance, [Lemma 11.6, Schilling and Partzsch [SR14]], which relies on a Fourier expansion, or the references pointed out in the footnote in [p. 80, Billingsley [Bl68]]. A different approach, which is connected to the proof of Theorem 3 below, is pointed out in [10, p. 273, [Sp76]].

3.3 An elementary proof approach to Theorem 3

Theorem 7. *Let τ_b and T be as in Theorem 3 and let also $S = \inf\{t : W(t) = 1\}$ and $\sigma_b = \inf\{t : S_t = b\}$, $b \geq 1$. We have that*

$$\lim_{b \rightarrow \infty} \mathbb{E}(e^{-\theta \frac{\tau_b}{b^2}}) = \mathbb{E}(e^{-\theta T}) = \operatorname{sech} \sqrt{2\theta}, \quad (19)$$

where $\operatorname{sech} \theta = (\cosh \theta)^{-1} = \frac{2}{e^\theta + e^{-\theta}}$, and

$$\lim_{b \rightarrow \infty} \mathbb{E}(e^{-\theta \frac{\sigma_b}{b^2}}) = \mathbb{E}(e^{-\theta S}) = \exp(-\sqrt{2\theta}), \quad (20)$$

$\theta > 0$.

Proof of Theorem 3. By Proposition 4, it suffices to show (15) for $H(\cdot)$ as in (9) and as in (10). We may derive (15) from (19) by invoking the following expansion in series of simple functions

$$\operatorname{sech}\sqrt{2\theta} = \frac{\pi}{2} \sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{\theta + \frac{\pi^2}{8}(2n+1)^2}, \quad (21)$$

or, alternatively, the geometric series representation as follows

$$\operatorname{sech}\sqrt{2\theta} = \frac{2e^{-\sqrt{2\theta}}}{1 - (-1)e^{-2\sqrt{2\theta}}} = 2 \sum_{k \geq 0} (-1)^k e^{-(2k+1)\sqrt{2\theta}}. \quad (22)$$

The expansion in (21) leads to (15) with $H(\cdot)$ in the form (9), whereas (22) leads (by term-to-term inversion) to it with $H(\cdot)$ in the form (10); cf. [[Sp76], p. 273, 10]. \square

We state an immediate byproduct of Theorem 7. Let $f_S(\cdot)$ denote the probability density associated to S .

Corollary 8. $f_S(t) = \frac{1}{\sqrt{2\pi t^3}} \exp(-1/2t), t \geq 0$.

Proof of Corollary 8. We evoke the known fact that the Laplace transform in (20) may be inverted; cf., for instance, [§ 9, Chpt. 2, [RW93]]. \square

Proof of Theorem 7. The proofs of the left-hand-sides parts of (19) and (20) are omitted since for positive random variables convergence of Laplace transforms is equivalent to convergence in distribution, and thus the former is equivalent to (13), whereas the latter follows by an application of Donsker's Theorem (see, for instance, [Example 8.6.6, [Dr10]]). From this theorem we have that it thus suffices to show the remaining parts of the claim for the simple random walk, defined as follows. Let $S_t = \sum_{s=1}^t \zeta_s$, where $(\zeta_i : i \geq 1)$ uniformly distributed $\{-1, 1\}$ -valued independent random variables.

Proposition 9. Let $\lambda(z) = \frac{1 - \sqrt{1 - z^2}}{z}$, $z \in (0, 1)$. We have that

$$\mathbb{E}(z^{\tau_b}) = \frac{2}{\lambda^b(z) + \lambda^{-b}(z)}, \quad (23)$$

and that

$$\mathbb{E}(z^{\sigma_b}) = \lambda^b(z). \quad (24)$$

Proof of Proposition 9. We use the two statements following. Their proofs rely on symmetry and are simple and short, so we give them below after the proof Theorem 7..

Lemma 10. S_{τ_b} and τ_b are mutually independent.

Lemma 11. $\tau_b < \infty$, for all b , a.s..

Let $M_n^\theta = (\text{sech}\theta)^n e^{\theta S_n}$, $\theta > 0$. We have that M_n^θ is a product of independent, mean 1 random variables, and hence a martingale. By Lemma 10 and Doob's Optional Stopping Theorem (cf., for instance, [§ 10.10, Theorem (b), (ii), [W91]]), whose hypotheses are satisfied due to that $|M_{n \wedge \tau_b}| \leq e^{\theta b}$, since $\text{sech}\theta < 1$, and $\tau_b < \infty$ a.s., we have that

$$\mathbb{E}(\text{sech}\theta)^{\tau_b} = \text{sech}(\theta b). \quad (25)$$

By Doob's Optional Stopping Theorem, whose hypotheses are again satisfied due to that $|M_{n \wedge \sigma_b}| \leq e^{\theta b}$, and, by Lemma 11, $\sigma_b < \infty$ a.s., we have that

$$\mathbb{E}(\text{sech}\theta)^{\sigma_b} = \exp(-\theta b). \quad (26)$$

Setting $\text{sech}\theta = z$ and since $z \in (0, 1)$ and $\mathbb{E}(z)^{\sigma_b}, \mathbb{E}(z)^{\tau_b} \in [0, 1]$, yields $\theta = \ln(1/\lambda(z))$; substituting this in (25) and in (26) gives (23) and (24) respectively. Thus, the proof is complete. \square

Lemma 12. Let $\lambda(z)$ be as in Proposition 9. We have that

$$\lim_{b \rightarrow \infty} [\lambda(e^{-\theta/b^2})]^b = e^{-\sqrt{2\theta}}, \quad \theta > 0.$$

Proof of Lemma 12. Observe that¹⁷ $1 - \lambda(z) \sim \sqrt{2}\sqrt{1-z}$, as $z \rightarrow 1^-$, and hence

$$b \left(1 - \lambda(e^{-\frac{\theta}{b^2}})\right) \sim b\sqrt{2}\sqrt{1 - e^{-\frac{\theta}{b^2}}}, \quad \text{as } b \rightarrow \infty, \quad (27)$$

However, we have that

$$\lim_{b \rightarrow \infty} b\sqrt{2}\sqrt{1 - e^{-\frac{\theta}{b^2}}} = \sqrt{2\theta}, \quad (28)$$

since, by the Maclaurin series expansion, $1 - e^{-x} = \sum_{k \geq 1} \frac{(-1)^{k-1} x^k}{k!}$. Combining (27) and (28) gives

$$\lim_{b \rightarrow \infty} \left(1 - \left(1 - \lambda(e^{-\frac{\theta}{b^2}})\right)\right)^b = e^{-\sqrt{2\theta}},$$

due to that $\lim_{n \rightarrow \infty} \left(1 + \frac{a(n)}{n}\right)^n = e^w$, $w = \lim_{n \rightarrow \infty} a(n)$. The last display completes this proof. \square

To finish the proof, simply note that Lemma 12 yields the right-hand-side equality in (20) from (24), and that in (19) from (23) and an application of the algebraic limit theorem. \square

Proof of Lemma 10. Note that

$$\mathbb{P}(\tau_b = t, S_{\tau_b} = \pm b) = \mathbb{P}(\tau_b = t) - \mathbb{P}(\tau_b = t, S_{\tau_b} = \mp b),$$

and hence, by symmetry, $\mathbb{P}(\tau_b = t, S_{\tau_b} = \pm b) = \frac{1}{2}\mathbb{P}(\tau_b = t)$, as required. \square

¹⁷as usual, we write $f(z) \sim g(z)$ as $z \rightarrow z_o$ to denote $\lim_{z \rightarrow z_o} \frac{f(z)}{g(z)} = 1$.

Proof of Lemma 11. By Kolmogorov's zero-one law (cf., for instance, [§ 4.11, Theorem, (ii), [W91]]) we have that $\limsup_{n \rightarrow \infty} S_n = c$, a.s., where c is a constant such that $c \in [-\infty, \infty]$. Clearly, it suffices to show that $c = +\infty$. Since also $\limsup_{n \rightarrow \infty} S_{n+1} = c$, a.s., we have that if $c < \infty$, then $\zeta_1 = 0$ a.s., and hence, by contradiction, $|c| = \infty$. Finally, if $c = -\infty$ a.s., then, by symmetry, $S'_n := -S_n \stackrel{d}{=} S_n$, and hence $\liminf_{n \rightarrow \infty} (S_n) = -\limsup_{n \rightarrow \infty} (-S_n) = +\infty$, which leads to the contradiction, $+\infty \leq -\infty$, and the proof is complete. \square

Remark 7. Other proofs of Theorem 7 rely on Donsker's theorem and typically deal directly with Brownian motion to derive first the right-hand-side equalities in (19) and in (20). A proof of Theorem 7 may be thus shown by relying on applications of Doob's Optional Stopping Theorem for so-called exponential martingales, see for instance, [Proposition 3.7, Chpt. II, Revuz and Yor [RY94]], or [Theorem 8.5.7, Durrett [Dr10]]. Another, heuristic proof approach to (20) can be found in [§ 13.7, Breiman [Br92]].

4 Proofs of the main results

4.1 Two preparatory lemmas

In this section we give two preparatory results, which we use in the proof of Proposition 16 below. The first one is a coupling result regarding the simple N -dimensional random walk which evokes a simple realization associated to its description in (2), see for instance, [§ 1.2.4, [St05]]. To state it, let $(\mathbf{Z}_t : t \geq 0)$ be the simple N -dimensional random walk and recall the definitions of \tilde{T}_r and \tilde{M}_t from (3) and (4) respectively.

Lemma 13. *Let $\{(Z_{n,t}, t \geq 1) : n = 1, \dots, N\}$ be a collection of independent simple one-dimensional random walks. Let $\tau_{n,r} = \inf\{t : |Z_{n,t}| > r\}$, $n = 1, \dots, N$. Let also $m_{n,t} = \max_{1 \leq s \leq t} |S_{n,t}|$. We can define $(\mathbf{Z}_t : t \geq 0)$ on the same probability space with $\{(Z_{n,t}, t \geq 1) : n = 1, \dots, N\}$, such that the following hold:*

$$\tilde{T}_r \leq \sum_{n=1}^N \tau_{n,r} - (N-1), \quad (29)$$

and

$$\tilde{M}_t \leq \sum_{n=1}^N m_{n,t}, \quad (30)$$

almost surely.

Proof. We obtain a realization of $(\mathbf{Z}_t : t \geq 1)$ as follows. The value of $\mathbf{Z}_{t+1} - \mathbf{Z}_t$ is decided by first choosing one of the N coordinates uniformly at random, and then deciding whether it is to be $\{+1, -1\}$. With this in mind, let $(d_s, s \geq 1)$ be a $\{1, \dots, N\}$ -valued independent and uniformly distributed collection of random variables, which is independent of $\{(Z_{n,t}, t \geq 1) : n = 1, \dots, N\}$. Letting $K_{n,t} = \sum_{s=1}^t I(d_s = n)$, $n = 1, \dots, N$, we have that

$$\mathbf{Z}_t \stackrel{d}{=} (Z_{K_{1,t}}, \dots, Z_{K_{N,t}}). \quad (31)$$

Let $\delta = \{i : |\mathbf{Z}_{\tilde{T}_r}(i)| > r\}$. From (31) we have that

$$\tilde{T}_r = \tau_{i,r} + K_{j,\tilde{T}_r} \text{ on } \{\delta = i\} \quad (32)$$

for all $i \neq j$, however

$$K_{j,\tilde{T}_r} \leq \tau_j - 1 \text{ on } \{\delta = i\}, \quad (33)$$

for all $i \neq j$, by contradiction to the definition of δ . Therefore, (32) and (33) imply (29).

Let $(\mathbf{S}_t : t \geq 1)$ be the concatenation $\mathbf{S}_t = (Z_{1,t}, \dots, Z_{N,t})$. Let $M_t = \max_{1 \leq s \leq t} |\mathbf{S}_s|$ and let also $\tilde{m}_{n,t} = \max_{1 \leq s \leq t} |Z_{K_{n,t}}|$. From (31) because $K_{n,t} \leq t$, for all n and t , we have that, for every n ,

$$\tilde{m}_{n,t} \leq m_{n,t} \text{ for all } t,$$

almost surely, and hence $\tilde{M}_t := \max_{n=1,\dots,N} \tilde{m}_{n,t} \leq \max_{n=1,\dots,N} m_{n,t} =: M_t$, which implies (30), and the proof is complete. \square

Lemma 14. *Let $S_t = \sum_{s=1}^t \zeta_s$ be the simple one-dimensional random walk. Let also $\tau_b = \inf\{t : |S_t| \geq b\}$ and $m_t = \max\{|S_1|, \dots, |S_t|\}$. We have that*

$$\left\{ \left(\frac{\tau_b}{b^2} \right)^p, b \geq 1 \right\} \text{ is uniformly integrable for all } p \in \mathbb{N}, \quad (34)$$

and that

$$\left\{ \left(\frac{m_t^r}{t^{r/2}} \right), t \geq 1 \right\} \text{ is uniformly integrable for all } r \in \mathbb{N}, \quad (35)$$

Proof. For both parts we will verify the assumptions of the following general result, which corresponds to the reverse part of [Theorem 5.4, Chpt. 1, [Bl68]].

Lemma 15. *Let $(X_n; n \geq 0)$ be a sequence of non-negative and integrable random variables. Suppose that $X_n \xrightarrow{d} X$ and that $\mathbb{E}(X_n) \rightarrow \mathbb{E}(X)$, as $n \rightarrow \infty$, as well as that $\mathbb{E}(X) < \infty$. Then $(X_n; n \geq 0)$ are uniformly integrable.*

From Theorem 3 the continuous mapping theorem yields

$$\left(\frac{\tau_b}{b^2} \right)^p \xrightarrow{d} T^p \text{ as } b \rightarrow \infty, \quad (36)$$

where the distribution of T^p is such that $\Phi_p(x) = \mathbb{P}(T^p \leq x) = 1 - H(x^{1/p})$, and H is as in (9). From a simple computation from the density function of T^p , we have that

$$\mathbb{E}(T^p) = p! \cdot \frac{\pi}{2} \left(\frac{8}{\pi^2} \right)^{(p+1)} \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{2k+1} \right)^{2p+1},$$

cf. [Proposition 3, § 23, [Sp76]], and hence

$$\mathbb{E}(T^p) < \infty. \quad (37)$$

Furthermore, if $L(\theta) = \mathbb{E}(e^{-\theta \frac{\tau_b}{b^2}})$, letting $L^{(p)}(\theta)$ denoting its derivative of the p -th order, we have

$$\mathbb{E} \left(\frac{\tau_b}{b^2} \right)^p = (-1)^p \cdot \lim_{\theta \rightarrow 0} L^{(p)}(\theta)$$

and thus, from [(23), Proposition 9] we get that the integrability condition of Lemma 15 is granted, which is that

$$\mathbb{E} \left(\frac{\tau_b}{b^2} \right)^p < \infty, \quad (38)$$

for all $b \geq 1$. In addition, evoking [Propositions 3 and 6, § 23, [Sp76]] we have that

$$\mathbb{E} \left(\frac{\tau_b}{b^2} \right)^p \rightarrow \mathbb{E}(T^p), \text{ as } b \rightarrow \infty. \quad (39)$$

Thus, from (36), (37), (38), and (39) the assumptions of Lemma 15 are satisfied, hence yielding (34).

For the second part, evoking the Lemma in [p. 69, [Bl68]] gives

$$\mathbb{P} \left(\frac{m_t}{\sqrt{t}} \geq x \right) \leq 2 \mathbb{P} \left(\frac{2|S_t|}{\sqrt{t}} \geq x \right), \quad (40)$$

for all $x > 2\sqrt{2}$. Letting $\bar{\Theta}_{t,r}(x) = \mathbb{P} \left(\frac{m_t^r}{t^{r/2}} \geq x \right)$ and $\bar{\Delta}_{t,r}(x) = \mathbb{P} \left(\frac{2|S_t|^r}{t^{r/2}} \geq x \right)$, (40) yields that

$$\bar{\Theta}_{t,r}(x) \leq 2\bar{\Delta}_{t,r}(x), \text{ for all } x > (2\sqrt{2})^r, \quad (41)$$

$r > 0$. Further, we have that

$$\left\{ \left(\frac{2|S_t|}{\sqrt{t}} \right)^p, t \geq 1 \right\} \text{ is uniformly integrable for all } p \in \mathbb{N}, \quad (42)$$

and, from the definition of uniform integrability, (42) and (41) yield (35).

To complete the proof it remains to show (42), for which we will again check the hypotheses of Lemma 15. We have that

$$\mathbb{E} \left(\frac{|S_t|}{\sqrt{t}} \right)^p \rightarrow \mathbb{E}|N|^p, \text{ as } t \rightarrow \infty, \quad (43)$$

for all $p \in \mathbb{N}$, where N denotes a standard normal. To derive (43) note that, for all $p \geq 2$, this is a direct consequence of the second part of the statement in [Theorem 7.5.1, [G12]], since we have that $\mathbb{E}(|\zeta_i|^p) = 1$. Whereas the case $p = 1$ follows from the central limit theorem, cf. Theorem 7.1.1 in [G12] and checking the general conditions for convergence in distribution to extend to convergence of moments, cf. for instance Theorem 5.5.1 in [G12]. To do this simply note that $\mathbb{E}(|\zeta_i|) = 1$, and hence that $(|\zeta_i| : i \geq 1)$ are uniformly integrable since these are uniformly bounded by an integrable random variable, cf. for instance, Theorem 5.4.4 in [G12]. Furthermore, from the central limit theorem and the continuous mapping theorem, cf. Theorem 5.10.4 in [G12], we have that the convergence in distribution analogue of (43) holds. In addition, we have that $\mathbb{E} \left(\frac{|S_t|}{\sqrt{t}} \right)^p < \infty$ from a consequence of the Marcinkiewicz–Zygmund inequalities, Corollary 3.8.2 in [G12], and that $\mathbb{E}|N|^p < \infty$. From these facts we have that (42) follows, and the proof is complete. \square

Remark 8. We note that an alternative route to show (42) may be concocted by consulting the proof of [Theorem 7.5.1, [G12]]; from the first display there, we have that (42) holds for all $p \geq 2$ since $\mathbb{E}(|\zeta_i|^p) = 1$, whereas (42) for $p = 1$ follows again from this fact by a simple criterion for uniform integrability, cf. for instance [Theorem 5.4.2, [G12]].

4.2 Proofs of Theorem 1 and Proposition 2

Proposition 16. *We have that*

$$\left\{ \left(\frac{\widetilde{T}_r^p}{r^{2p}}, r \geq 1 \right) \right\} \text{ is uniformly integrable for all } p \geq 1, \quad (44)$$

and further, that

$$\left\{ \left(\frac{\widetilde{M}_t^p}{t^{p/2}}, t \geq 1 \right) \right\} \text{ is uniformly integrable for all } p \geq 1. \quad (45)$$

Proof. We will need the following generic Lemma.

Lemma 17. *If $\{(X_{n,k}^p)_{k=1}^\infty; n = 1, \dots, N\}$ is a collection of uniformly integrable sequences of positive random variables for every $p \in \mathbb{N}$, then $\left(\left(\sum_{n=1}^N X_{n,k}\right)^p\right)_{k=1}^\infty$ is uniformly integrable for every $p \in \mathbb{N}$.*

Proof of Lemma 17. The proof proceeds by induction in N . Clearly the statement holds for $N = 1$. We assume that it holds for $N - 1$. Let $Y_k := \sum_{n=1}^{N-1} X_{n,k}$ and $W_k := X_{N,k}$. Since finite sums of uniformly integrable random variables are uniformly integrable (cf., for instance, Theorem 5.4.6 in [G12]), an application of binomial theorem gives that it suffices to show that $cY_k^a W_k^b$ are uniformly integrable for all a and b non-negative integers such that $a + b = p$, and where c can be taken to be $c = 1$ from the definition of uniform integrability without loss of generality. From an application of the Hölder inequality (cf., for instance, Theorem 5.4.7 in [G12]), we have that it suffices to show that Y_k^{2a} and W_k^{2b} are uniformly integrable. However, this holds for the former sequence by the induction hypothesis, and for the latter by the assumptions of the theorem. \square

From [(29), Lemma 13] we have that

$$\left(\frac{\widetilde{T}_r}{r^2} \right)^p \leq \left(\sum_{n=1}^N \frac{\tau_{n,r}}{r^2} \right)^p \quad (46)$$

almost surely. Further, from [(30), Lemma 13] we have that

$$\left(\frac{\widetilde{M}_t}{\sqrt{t}} \right)^p \leq \left(\sum_{n=1}^N \frac{m_{n,t}}{\sqrt{t}} \right)^p \quad (47)$$

almost surely. From Lemma 17, we hence have that (44) follows from (46) due to [(34); Lemma 14], whereas (45) follows from (47) due to [(35); Lemma 14]. \square

Proof of Proposition 2. Let $(\mathbf{W}_t : t \in \mathbb{R}_+)$ be standard Brownian motion, covariance matrix I , in \mathbb{R}^N , and let also $M_t = \sup_{0 \leq s \leq t} |\mathbf{W}_s|$. From, for instance, Lemma 3.4.1 in [LL10], we have that, if \mathbf{W}'_t is Brownian motion in \mathbb{R}^N with covariance matrix Σ , then

$$\mathbf{W}'_t = \sum_{n=1}^N v_n W_{n,t} \quad (48)$$

provided $\Sigma = VV^T$ and $V = [v_1, \dots, v_N]$, and where $(W_{n,t}^i; n = 1, \dots, N)$ are standard mutually independent Brownian motions in \mathbb{R} . Hence, note that \mathbf{W}_t is of the form $\mathbf{W}_t = (W_{1,t}, \dots, W_{N,t})$ and, letting $M_t = \sup_{0 \leq s \leq t} |\mathbf{W}_s|$ and $m_{n,t} = \sup_{s \in [0,t]} |W_{n,t}|$, we have that $(m_{n,t} : n = 1, \dots, N)$ are independent and identically distributed according to Γ_t in Corollary 6. Hence, we have that

$$\begin{aligned} \mathbb{P}(M_t < x) &= \prod_{n=1}^N \mathbb{P}(m_{n,t} < x) \\ &= \left(H\left(\frac{t}{x^2}\right) \right)^N. \end{aligned} \quad (49)$$

Note that, from another application of (48), we further have that

$$\widetilde{\mathbf{W}}_t \stackrel{d}{=} \frac{1}{N} \mathbf{W}_t, \quad (50)$$

from which it follows that $\widetilde{M} \stackrel{d}{=} \frac{1}{N} M_1$, which yields (8) from (49).

We now prove (7). Letting $\widetilde{M}_t = \sup_{0 \leq s \leq t} |\widetilde{\mathbf{W}}_s|$, we have that

$$\begin{aligned} \mathbb{P}(\widetilde{T} < t) &= \mathbb{P}(\widetilde{M}_t > 1) \\ &= 1 - \mathbb{P}\left(\frac{1}{N} M_t < 1\right), \end{aligned} \quad (51)$$

where the first line is by coupling and the next comes from another application of (50). Thus, (7) follows from (51) and (49). \square

Proof of Theorem 1. We will first show the following statement we need to use.

Lemma 18.

$$\frac{\widetilde{T}_r^p}{r^{2p}} \xrightarrow{d} \widetilde{T}^p \quad \text{as } r \rightarrow \infty. \quad (52)$$

and that

$$\frac{1}{t^{p/2}} \widetilde{M}_t^p \xrightarrow{d} \widetilde{M}^p \quad \text{as } t \rightarrow \infty, \quad (53)$$

Proof. Let C_N be all continuous $\phi : [0, \infty) \rightarrow \mathbb{R}^N$, equip with the usual metric ρ , so that $\rho(\phi_n, \phi_0) \rightarrow 0$, as $n \rightarrow \infty$, if and only if, for any k ,

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq k} |\phi_n(t) - \phi_0(t)| = 0.$$

Let $\mathbf{W}^{(n)}(s, \omega) = \frac{1}{\sqrt{n}} \mathbf{Z}_{[ns]}(\omega)$, let also $\mathbf{W}(t)$ be N -dimensional Brownian motion covariance matrix $\Sigma = \frac{1}{N} I$, and further let $\Psi : C_N \rightarrow \mathbb{R}$ be any function which is continuous. From, for instance, [Theorem 3.5.1, [LL10]], we have that

$$\Psi(\mathbf{W}^{(n)}) \Rightarrow \Psi(\mathbf{W}), \quad \text{as } n \rightarrow \infty, \quad (54)$$

in C_N .

Let $\Psi_1(\mathbf{x}) = \inf\{s : |\mathbf{x}(s)| > 1\}$ and let $M_n = \inf\{t : |\mathbf{Z}_t| > \sqrt{n}\}$. Since Ψ_1 is a.s. continuous and, by linear interpolation,

$$\inf\{ns : |\mathbf{Z}_{[ns]}| \geq \sqrt{n}\} = \inf\{s : |\mathbf{Z}_s| \geq \sqrt{n}\},$$

for all n , (54) gives

$$\frac{M_n}{n} \xrightarrow{d} \tilde{T}, \text{ as } n \rightarrow \infty,$$

so that (52) follows by an application of the continuous mapping theorem.

Let $\Psi_2(\mathbf{x}) = \sup_{0 \leq s \leq 1} |\mathbf{x}(s)|$. Since Ψ_2 is a.s. continuous and by linear interpolation we have that

$$\sup_{0 \leq s \leq 1} \frac{|\mathbf{Z}_{[ns]}|}{\sqrt{n}} = \sup_{0 \leq j \leq n} \frac{|\mathbf{Z}_j|}{\sqrt{n}},$$

for all n , (54) gives

$$\sup_{0 \leq j \leq n} \frac{|\mathbf{Z}_j|}{\sqrt{n}} \xrightarrow{d} \widetilde{M}.$$

which yields (53) by an application of the continuous mapping theorem, and the proof is complete. □

To complete the proof, note that the limiting distribution associated to (52) and (53) are given by [11, Proposition 2] and by [12, Proposition 2] respectively. Hence, by a general result, see for instance [Theorem 5.5.9, [G12]], we have that (5) follows from (52) combined with [(44), Proposition 16], and that (6) follows from (53) combined with [(45), Proposition 16]. Thus, the proof is complete. □

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References

- [Bl68] BILLINGSLEY, P. (1968). *Convergence of probability measures*. John Wiley & Sons.
- [Br92] BREIMAN, L. (1992). *Probability*, SIAM.
- [Dn51] DONSKER, M. (1951). An invariance principle for certain probability limit theorems. AMS. 690
- [Dr10] DURRETT, R. (2010). *Probability: theory and examples*. Cambridge university press
- [EK46] ERDÖS, P., AND KAC, M. (1946). On certain limit theorems of the theory of probability. Bulletin of the American Mathematical Soc., 52(4), 292-302.
- [E10] ETHIER, S.N. (2010). *The doctrine of chances: probabilistic aspects of gambling*. Springer.
- [Fl68] FELLER, W. (1968). *An introduction to probability theory and its applications*. Vol. 1, John Wiley & Sons.

- [Fl70] FELLER, W. (1970). *An introduction to probability theory and its applications*. Vol. 2, John Wiley & Sons.
- [G12] GUT, A. (2012). *Probability: a graduate course*. Springer. (First Edition).
- [G09] GUT, A. (2009) *Stopped random walks*. Springer.
- [Kll97] KALLENBERG, O. (1997). *Foundations of modern probability*.
- [KT81] KARLIN, S., AND TAYLOR, H. (1981). *A second course in stochastic processes*.
- [KP02] KMET, A., AND PETKOVŠEK, M. (2002). Gambler’s ruin problem in several dimensions. *Advances in applied Mathematics* 28.2: 107-118.
- [Kl31] KOLMOGOROV, A. (1931). Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung. *Math. Ann.* 104. 415-458.
- [Kl33] KOLMOGOROV, A. (1933). Sulla determinazione empirica di una legge di distribuzione.
- [Ll] LALLEY, S. *One-dimensional random walks*. Lecture notes (Unpublished).
- [Lw10] LAWLER, G.F. (2010) *Random walk and the heat equation*. Vol. 55. American Mathematical Soc..
- [LL10] LAWLER, G.F., AND LIMIC, V. (2010) *Random walk: a modern introduction*. Vol. 123. Cambridge University Press.
- [Ls05] LESIGNE, E. (2005). *Heads or tails: an introduction to limit theorems in probability*. American Mathematical Soc..
- [Lv48] LÉVY, P.. (1948). *Processus Stochastiques et Mouvement Brownien*. Gauthier-Villars, Paris.
- [MP10] MÖRTERS, P., AND PERES, Y. (2010). *Brownian motion*. Cambridge University Press.
- [OZ94] ORR, C., AND ZEILBERGER, D. (1994). A computer algebra approach to the discrete Dirichlet problem. *Journal of Symbolic Computation*, 18(1), 87-90.
- [R90] RÉVÉSZ, P. (1990) *Random walk in random and non-random environments*. World Scientific.
- [RY94] REVUZ, D., AND YOR, M. (1994). *Continuous martingales and Brownian motion*. Springer.
- [RW93] ROGERS, L.C.G. AND WILLIAMS, D. (1993). *Diffusions, Markov processes, and martingales, Vol. 1.*, John Wiley & Sons.
- [SR14] SCHILLING, R., AND PARTZSCH, L. (2014). *Brownian motion: an introduction to stochastic processes*.
- [Sp76] SPITZER, F. (1976). *Principles of random walk*. Springer.
- [St05] STROOCK, D.W. (2005). *An Introduction to Markov processes*. Springer.
- [W91] WILLIAMS, D. (1991). *Probability with martingales*. Cambridge University Press.

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